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ON THE ISOTROPIC GROUP OF A HOMOGENEOUS POLYNOMIAL

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ABSTRACT. Let G be the linear group leaving a homogeneous polynomial of degree k fixed. The author shows that either the polynomial is a polynomial in fewer than the assigned number of variables or that the $(k-1)$st prolongation of G is 0. The author also shows that this result is optimal.

Let $P^k[X_1, \cdots, X_n]$ be the space of homogeneous polynomials in $X_1, \cdots, X_n$ of degree k. Note that the linear group $\mathrm{GL}(n)$ acts on $P^k[X_1, \cdots, X_n]$. If $A = (a_i^j) \in \mathrm{GL}(n)$ and $f(X_1, \cdots, X_n) \in P^k[X_1, \cdots, X_n]$, we denote the action of A on f by Af where

$$(1) \qquad Af(X_1, \cdots, X_n) = f\left(\sum_{j=1}^n a_1^j X_j, \cdots, \sum_{j=1}^n a_n^j X_j\right).$$

The Lie algebra $\mathrm{gl}(n)$ of $\mathrm{GL}(n)$ is the set of $n \times n$ matrices which also acts on $P^k[X_1, \cdots, X_n]$. If $a = (a_i^j) \in \mathrm{gl}(n)$, $f \in P^k[X_1, \cdots, X_n]$ then

$$(2) \qquad a \cdot f(X_1, \cdots, X_n) = \sum_{i,j}^n \frac{\partial f}{\partial X_i} a_i^j X_j.$$

This is obtained by writing $A = \exp(ta)$ and differentiating (1) with respect to t.

Let G be the Lie sugbroup of $\mathrm{GL}(n)$ which leaves $f \in P^k[X_1, \cdots, X_n]$ invariant, and g the Lie algebra of G. By formula (2) we have the following.

Lemma. *The Lie algebra g of G consists of those matrices (a_i^j) for which* $\Sigma_{i,j}^n \partial f/\partial X_i \, a_i^j X_j = 0$.

For each element $a = (a_i^j) \in \mathrm{gl}(n)$ we can think of $(A^1, \cdots, A^n)$ as an n-tuple of homogeneous polynomials of first order, where $A^i = \Sigma_{j=1}^n a_i^j X_j$. From now on we identify $\mathrm{gl}(n)$ with the space of n-tuple of homogeneous polynomials of first order. So we can also think of the subspace g of $\mathrm{gl}(n)$ as a subspace of the above space.

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For any Lie subalgebra g (see [3]) of $gl(n)$ we define the *first prolongation* $g^{(1)}$ of g as the set of all n-tuples. $(A^1(X), \cdots, A^n(X))$, where A^i is a homogeneous polynomial of degree two in $X_1, \cdots, X_n$ and all the first derivatives, $i = 1, \cdots, n$, $(\partial A^1/\partial X_i, \cdots, \partial A^n/\partial X_i)$ lie in g. Similarly we define the kth prolongation $g^{(k)}$ of g as the set of all n-tuples $(A^1(X), \cdots, A^n(X))$ where $A^i(X)$ is a homogeneous polynomial of degree $k+1$ in $X_1, \cdots, X_n$, and all the kth derivatives $(\partial^k A^1/\partial X^\alpha, \ldots, \partial^k A^n/\partial X^\alpha)$, $|\alpha| = k$, lie in g.

Definition. The algebra g is of finite type if $g^{(k)} = 0$ for some k (and, therefore for all large k) G is said to be of type k if $g^{(k-1)} \neq 0$ and $g^{(k)} = 0$.

Definition. $f(X)$ is a nonsingular homogeneous polynomial if f cannot be written as a polynomial in fewer than n variables.

We have the following result:

Theorem. *If $f \in P^k[X_1, \cdots, X_n]$ and nonsingular then* $g^{(k-1)} = 0$.

Proof. If $(A^1(X), \cdots, A^n(X)) \in G^{(k-1)}$, by definition

$$(\partial^{k-1}A^1/\partial X^\alpha, \ldots, \partial^{k-1}A^n/\partial X^\alpha), \qquad |\alpha| = k-1, \in g.$$

then

$$\sum_{i=1} \frac{\partial f}{\partial X_i} \frac{\partial^{k-1}}{\partial X^\alpha} A^i = 0 \quad \text{for all } |\alpha| = k-1.$$

By repeated application of Euler's identity for homogeneous functions we can rewrite this as

$$\sum_{i=1}^{n} \frac{\partial^{k-1}}{\partial X^\alpha} \left(\frac{\partial f}{\partial X_i}\right) A^i = 0 \quad \text{for all } |\alpha| = k-1.$$

However $\partial^{k-1}/\partial X^\alpha \, (\partial f/\partial X_i)$ are constants, so choosing any X_0 such that the $A^i(X_0)$ are not all zero, and setting $b_i = A^i(X_0)$ we get

$$\sum \frac{\partial^{k-1}}{\partial X^\alpha} \left(\frac{\partial f}{\partial X_i}\right) b_i = 0 \quad \text{for all } |\alpha| = k-1.$$

This implies $\sum b_i \partial f/\partial X_i = 0$, in other words f can be written as a polynomial in $n-1$ variables which contradicts the hypotheses. Q.E.D.

In what follows we construct some examples which show that the above theorem is the best possible result we can get. Let

$$f(X_1, X_2, X_3, X_4, X_5) = \begin{vmatrix} X_3 & X_4 & X_5 \\ X_1 & X_2 & 0 \\ 0 & X_1 & X_2 \end{vmatrix} = X_3X_2^2 - X_4X_1X_2 + X_5X_1^2.$$

It is easily seen that f cannot be written as a polynomial fewer that 5 variables. Let $(A^1, A^2, A^3, A^4, A^5)$ be a 5-tuple of homogeneous polynomials of second degree where $A^1 = 0$, $A^2 = 0$, $A^3 = X_1^2$, $A^4 = 2X_1X_2$, $A^5 = X_2^2$. Then

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i} \frac{\partial A^i}{\partial X_j} = 0 \quad \text{for all } j = 1, 2, 3, 4, 5.$$

This implies $(A^1, A^2, A^3, A^4, A^5) \in g^{(1)}$, so $g^{(1)}$ is nontrivial.

Let $f \in P^{k-2}[X_1, X_2, \cdots, X_k]$ be defined as follows:

$$\begin{array}{ll} A^1 = 0, & A^4 = X_1^{k-4}X_2, \\ A^2 = 0, & A^i = X_1^{k-i}X_2^{i-3}, \\ A^3 = X_1^{k-3}, & A^k = X_2^{k-3}; \end{array}$$

$$f(X_1, X_2, \cdots, X_k) = \begin{vmatrix} \dfrac{\partial^{k-4}A^3}{\partial X_1^{k-4}} & \dfrac{\partial^{k-4}A^4}{\partial A_1^{k-4}} & \cdots & \dfrac{\partial^{k-4}A^k}{\partial X_1^{k-4}} \\ \dfrac{\partial^{k-4}A^3}{\partial X_1^{k-5}\partial X_2} & \dfrac{\partial^{k-4}A^4}{\partial X_1^{k-5}\partial X_2} & \cdots & \dfrac{\partial^{k-4}A^k}{\partial X_1^{k-5}\partial X_2} \\ \dfrac{\partial^{k-4}A^3}{\partial X_2^{k-4}} & \dfrac{\partial^{k-4}A^4}{\partial X_2^{k-4}} & \cdots & \dfrac{\partial^{k-4}A^k}{\partial X_2^{k-4}} \end{vmatrix}$$

It is easily seen that f is nonsingular and also

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i} \frac{\partial^{k-4}A^i}{\partial X^\alpha} = 0 \quad \text{for all } |\alpha| = k-4$$

so we have g^{k-4} of f is nontrivial.

Note that the first example of $g^{(1)} \neq 0$ occurs in dimension 5. Using a result of Gordan-Noether (see [1]) one can show that, for dimensions ≤ 4, $g^{(1)} = 0$ and, for dimension 5, $g^{(2)} = 0$.

The result above has interesting consequences for the problem of when a partial differential operator can be transformed to a constant coefficient operator see [2].

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